Određeni integrali

(75) $\int_0^8 \left(1+\sqrt{2x}+\sqrt[3]{x}\right) dx = \int_0^8 dx + \sqrt{2}\int_0^8 \sqrt{x}\, dx + \int_0^8 \sqrt[3]{x}\, dx =$

$= x\Big|_0^8 + \sqrt{2}\cdot \frac{2}{3}\sqrt{x^3}\Big|_0^8 + \frac{3}{4}\sqrt[3]{x^4}\Big|_0^8 =$

$= 8-0 + \frac{2\sqrt{2}}{3}\left(\sqrt{2}\cdot 2^4 - 0\right) + \frac{3}{4}\left(2^4-0\right) =$

$= 8 + \frac{64}{3} + 12 = \frac{124}{3}$

76) $\displaystyle\int_{-1}^{1} \frac{x}{\sqrt{5-4x}}\,dx = \left[\begin{array}{l} 5-4x = t \\ dx = -\frac{1}{4}dt \end{array} \quad \begin{array}{c|c|c} x & -1 & 1 \\ \hline t & 9 & 1 \end{array}\right] =$

$$= -\frac{1}{4}\int_{9}^{1} \frac{\frac{5-t}{4}}{\sqrt{t}}\,dt = -\frac{1}{16}\int_{9}^{1} \frac{5-t}{\sqrt{t}}\,dt =$$

$$= \frac{1}{16}\int_{1}^{9} \left(5\,t^{-\frac{1}{2}} - t^{\frac{1}{2}}\right) dt = \frac{5}{16}\,2\sqrt{t}\,\Big|_{1}^{9} - \frac{1}{16}\cdot\frac{2}{3}\sqrt{t^3}\,\Big|_{1}^{9} = \frac{1}{6}$$

---

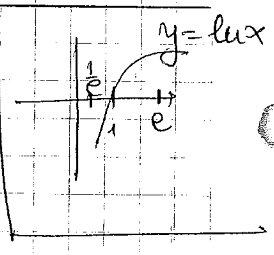
77) $\displaystyle\int_{0}^{2} f(x)\,dx, \quad f(x) = \begin{cases} x^2, & 0 \le x \le 1 \\ 2-x, & 1 < x \le 2 \end{cases}$

$$\int_{0}^{2} f(x)\,dx = \int_{0}^{1} f(x)\,dx + \int_{1}^{2} f(x)\,dx = \int_{0}^{1} x^2\,dx + \int_{1}^{2} (2-x)\,dx =$$

$$= \frac{x^3}{3}\Big|_{0}^{1} + 2x\Big|_{1}^{2} - \frac{x^2}{2}\Big|_{1}^{2} = \frac{5}{6}$$

---

78) $\displaystyle\int_{\frac{1}{e}}^{e} |\ln x|\,dx = \int_{\frac{1}{e}}^{1} |\ln x|\,dx + \int_{1}^{e} |\ln x|\,dx =$



$$= \int_{\frac{1}{e}}^{1} (-\ln x)\,dx + \int_{1}^{e} \ln x\,dx = \left[\begin{array}{l} U = \ln x \Rightarrow dU = \frac{1}{x}dx \\ V = \int dx = x \end{array}\right] =$$

$$= -\left(x\ln x\,\Big|_{\frac{1}{e}}^{1} - \int_{\frac{1}{e}}^{1} dx\right) + x\ln x\,\Big|_{1}^{e} - \int_{1}^{e} dx =$$

$$= -x\ln x\,\Big|_{\frac{1}{e}}^{1} + x\,\Big|_{\frac{1}{e}}^{1} + e\ln e - 1\ln 1 - x\,\Big|_{1}^{e} =$$

$$= -\frac{1}{e} + 1 - \frac{1}{e} + e - e + 1 = 2 - \frac{2}{e}$$

$\int R(x, \sqrt{a^2-x^2})\,dx \to x = a\sin t$ ili $x = a\cos t$

(79) $\displaystyle\int_0^1 \frac{x^2}{\sqrt{4-x^2}}\,dx = \left|\begin{array}{l} x = 2\sin t \\ dx = 2\cos t\,dt \\ \sin t = \frac{x}{2} \\ t = \arcsin\frac{x}{2} \end{array}\quad \begin{array}{c|c|c} x & 0 & 1 \\ \hline t & 0 & \frac{\pi}{6} \end{array}\right| =$

$\displaystyle = \int_0^{\frac{\pi}{6}} \frac{4\sin^2 t}{\sqrt{4-4\sin^2 t}} \cdot 2\cos t\,dt = 4\int_0^{\frac{\pi}{6}} \frac{\sin^2 t\cos t}{|\cos t|}\,dt =$ → oslobađamo se lako jer $0 - \pi/6 > 0$, ali PAŽLJIVO!

$\displaystyle = 4\cdot\int_0^{\frac{\pi}{6}} \frac{\sin^2 t\,\cancel{\cos t}}{\cancel{\cos t}}\,dt = 4\int_0^{\frac{\pi}{6}} \frac{1-\cos 2t}{2}\,dt =$

$\displaystyle = 2t\Big|_0^{\frac{\pi}{6}} - 2\cdot\frac{1}{2}\sin 2t\Big|_0^{\frac{\pi}{6}} = 2\cdot\frac{\pi}{6} - \frac{\sqrt{3}}{2} = \frac{\pi}{3} - \frac{\sqrt{3}}{2}$

---

(80) a) Dokazati da je: $\displaystyle\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x)$

- Neka je $F(x)$ primitivna funkcija funkcije $f(x)$.

$\displaystyle\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = \frac{d}{dx}\left(F(t)\Big|_a^x\right) = \frac{d}{dx}\left(F(x) - F(a)\right) =$

izvod const = 0

$= F'(x) = f(x)$ ubacimo u zagradu sa $\frac{d}{dx}$

b) Izračunati $\displaystyle\frac{d}{dx}\left(\int_x^b \sin t^2\,dt\right)$

Neka je $f(t) = \sin t^2$ i neka je F primitivna funkcija funkcije f.

$\displaystyle\frac{d}{dx}\left(\int_x^b \sin t^2\,dt\right) = \frac{d}{dx}\left(\int_x^b f(t)\,dt\right) =$

$\displaystyle = \frac{d}{dx}\left(F(t)\Big|_x^b\right) = \frac{d}{dx}\left(F(b) - F(x)\right) =$

$= 0 - F'(x) = -f(x) = -\sin x^2$

JORDI LABANDA

Površina ravnog lika

(81) Izračunati površinu figure ograničene krivom $y = x^2 + 2x + 2$ i pravama $y=0$, $x=2$ i $x=-3$.

$y = x^2 + 2x + 2 \rightarrow a=1;\ b=2;\ c=2$

$D = b^2 - 4ac = 4 - 4\cdot 2\cdot 1 = -4 < 0$ → kriva nema presjeka sa x osom

Tjeme $\left(-\frac{b}{2a}, -\frac{D}{4a}\right)$ $T\left(-\frac{2}{2}, -\frac{-4}{4}\right)$

$\boxed{T(-1, 1)}$

Presjek parabole i y ose $(x=0)$

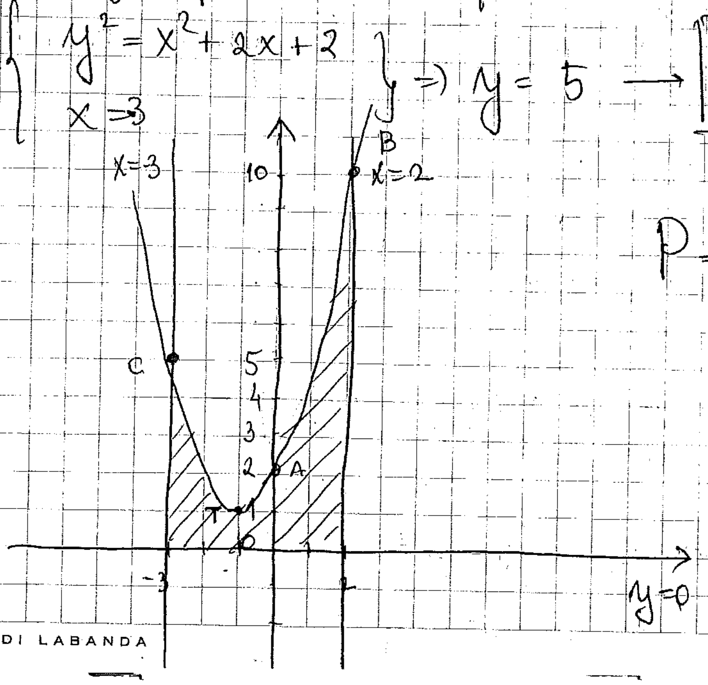
$\left\{\begin{array}{l} y = x^2 + 2x + 2 \\ x = 0 \end{array}\right\} \Rightarrow y = 2 \rightarrow \boxed{A(0, 2)}$

Presjek parabole i prave $x = 2$

$\left\{\begin{array}{l} y = x^2 + 2x + 2 \\ x = 2 \end{array}\right\} \Rightarrow y = 10 \rightarrow \boxed{B(2, 10)}$

Presjek parabole i prave $x = -3$

$\left\{\begin{array}{l} y^2 = x^2 + 2x + 2 \\ x = 3 \end{array}\right\} \Rightarrow y = 5 \rightarrow \boxed{C(-3, 5)}$



$$P = \int_{-3}^{2} (x^2 + 2x + 2)\,dx =$$

$$= \frac{x^3}{3}\Big|_{-3}^{2} + \frac{x^2}{1}\Big|_{3}^{2} + 2x\Big|_{3}^{2} =$$

$$= \frac{50}{30} = \frac{5}{3}$$

81. Izračunati površinu ograničenu krivom $y = x^2 - 1$, osom Ox i pravama $x = 0$ i $x = 2$.

$y = x^2 - 1 \rightarrow y = 0 \Leftrightarrow x = 1 \vee x = -1$

$\hookrightarrow$ presjek sa x osom ($y = 0$) $\rightarrow \boxed{A_1(-1, 0)}$ $\boxed{A_2(1, 0)}$

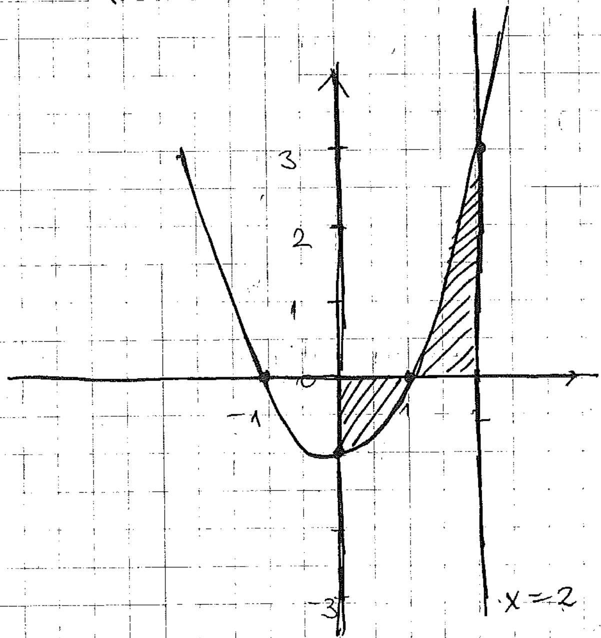
Presjek sa Oy osom ($x = 0$) → ovo će ujedno biti i tjeme

→ presjek sa Oy osom je ujedno i tjeme kada je $b = 0$

$$\begin{cases} y = x^2 - 1 \\ x = 0 \end{cases} \Rightarrow y = -1 \quad \boxed{T(0, -1)}$$

Presjek parabole i prave $x = 2$

$$\left.\begin{cases} y = x^2 - 1 \\ x = 2 \end{cases}\right\} \Rightarrow y = 3 \quad \boxed{B(2, 3)}$$



$$P = \int_0^2 |x^2 - 1|\, dx =$$

$$= -\int_0^1 (x^2 - 1)\, dx + \int_1^2 (x^2 - 1)\, dx$$

površina mora da bude pozitivna; $x^2 - 1$ na ovom djelu negativna fja

$$P = -\frac{x^3}{3}\Big|_0^1 + x\Big|_0^1 + \frac{x^3}{3}\Big|_1^2 - x\Big|_1^2 = 2$$

82) Izračunati površinu figure ograničene parabolom $y = x^2 + 4x$ i pravom $y = x + 4$.

Presjek parabole sa Ox osom ($y = 0$)

$$y = x(x+4)$$

$$\left\{\begin{array}{l} y = x^2 + 4x \\ y = 0 \end{array}\right\} \quad \begin{array}{l} x_1 = 0 \to y_1 = 0 \\ x_2 = -4 \to y_2 = 0 \end{array} \quad \boxed{A_1(0,0)} \quad \boxed{A_2(-4,0)}$$

Presjek parabole sa Oy osom ($x = 0$) → tjeme

$$\text{Tjeme}\left(-\frac{b}{2a}, -\frac{D}{4a}\right) \qquad T\left(\frac{-4}{2}, -\frac{16}{4}\right)$$

$b = 4 \quad c = 0$

$a = 1 \quad D = b^2 - 4ac = b^2$ $\qquad \boxed{T(-2, -4)}$

Presjek parabole sa pravom $y = x + 4$

$$\left\{\begin{array}{l} y = x^2 + 4x \\ y = x + 4 \end{array}\right\} \quad \begin{array}{l} x + 4 = x^2 + 4x \\ x^2 + 3x - 4 = 0 \end{array} \to x_{1,2} = \frac{-3 \pm \sqrt{9+16}}{2} = \frac{-3 \pm 5}{2} = \left\langle\begin{array}{l} 1 \\ -4 \end{array}\right.$$

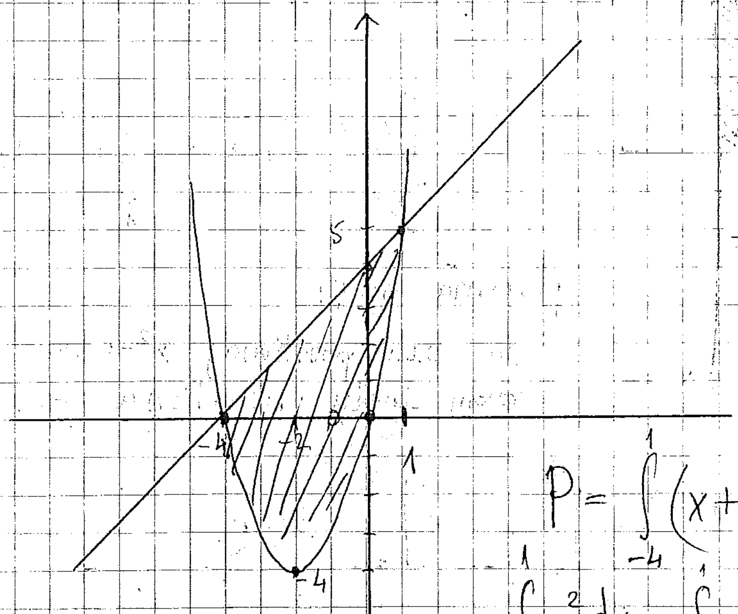
$y_1 = 5 \quad y_2 = 0$

$\boxed{B_1(-4,0)} \quad \boxed{B_2(1,5)}$

$y = x + 4$

$y = 0 \Rightarrow x = -4$

$x = 0 \Rightarrow y = 4$



→ na intervalu -4 do 1 ⇒ prava iznad parabole

$$\Rightarrow P = \int_{-4}^{1} \text{prava} - \text{parabola}$$

$$P = \int_{-4}^{1} (x + 4 - x^2 - 4x)\,dx = \int_{-4}^{1} (-x^2 - 3x + 4)\,dx =$$

$$= -\int_{-4}^{1} x^2\,dx - 3\int_{-4}^{1} x\,dx + \int_{-4}^{1} dx = -\frac{x^3}{3}\Big|_{-4}^{1} - \frac{3}{2}x^2\Big|_{-4}^{1} + 4x\Big|_{-4}^{1}$$

$$= -\frac{1}{3} - \frac{64}{3} - \frac{3}{2}(1 - 16) + 4 + 16 = \frac{125}{6}$$

83) Izračunati površinu ograničenu krivom

$y^2 = 2x+1$ i pravom $y = x-1 \to \boxed{x = y+1}$

$y^2 = 2x+1$

$2x = y^2 - 1$

$\boxed{x = \frac{1}{2}y^2 - \frac{1}{2}}$

$a = \frac{1}{2}$

$b = 0$

$c = -\frac{1}{2}$

$x = ay^2 + by + c$

$T = \left(-\frac{D}{4a}, -\frac{b}{2a}\right)$

$D = b^2 - 4ac \to$ ili $\supset$ ili $\subset$

$\to$ pitamo da li siječe y osu

Presjek sa Oy osom $(x=0)$

$x = 0 \iff \frac{1}{2}y^2 - \frac{1}{2} = 0 \to \frac{1}{2}y^2 = \frac{1}{2} \to y = \pm 1$

$\boxed{A_1(0, 1)}$ $\boxed{A_2(0, -1)}$

Presjek sa Ox osom $\to y = 0$

$T\left(-\frac{1}{2}, 0\right)$

$y^2 = 2x+1 \to 2x = -1 \Rightarrow x = -\frac{1}{2}$

$x = \frac{1}{2}y^2 - \frac{1}{2}$; $y = 0 \Rightarrow x = -\frac{1}{2}$ nema b $\Rightarrow$ tjeme presjek sa Ox osom

Presjek parabole i prave $y = x-1$

$$\begin{cases} y^2 = 2x+1 \\ y = x-1 \end{cases}$$

$(x-1)^2 = 2x+1$

$x^2 - 2x + \cancel{1} = 2x + \cancel{1}$

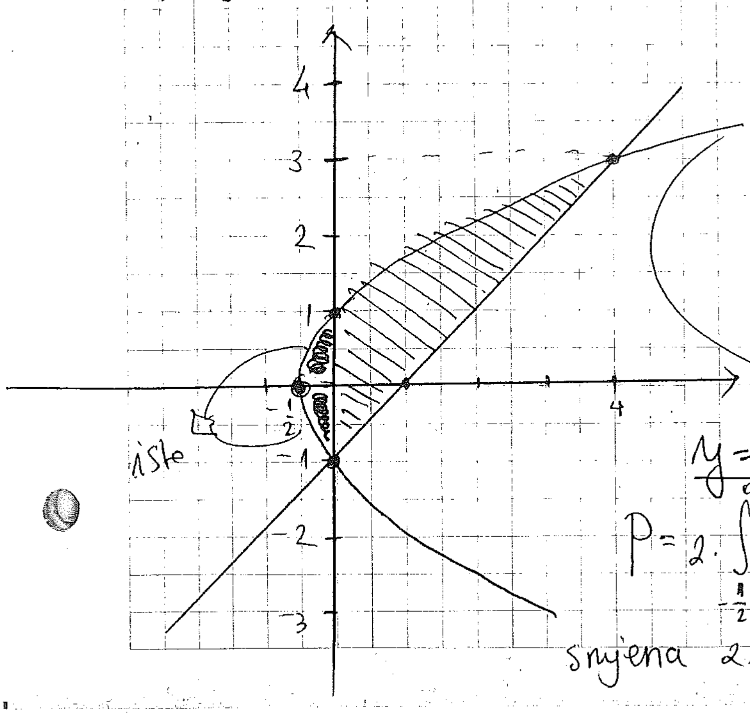
$x^2 - 4x = 0$

$x(x-4) = 0 \iff x = 0 \lor x = 4$

$y = -1$ $y = 3$

$B_1(0, -1)$ $B_2(4, 3)$

$x = f(y)$



$$P = \int_{-1}^{3}\left(y + 1 - \frac{1}{2}y^2 + \frac{1}{2}\right)dy =$$

$$= -\frac{1}{2}\frac{y^3}{3}\Big|_{-1}^{3} + \frac{y^2}{2}\Big|_{-1}^{3} + \frac{3}{2}y\Big|_{-1}^{3} = \frac{16}{3}$$

$y = f(x)$

$$P = 2\cdot\int_{-\frac{1}{2}}^{0}\sqrt{2x+1}\,dx + \int_{0}^{4}\left(\sqrt{2x+1} - (x-1)\right)dx$$

smjena $2x+1 = t$

(84) U presječnim tačkama prave $x - y + 1 = 0$ i parabole $y = x^2 - 4x + 5$ povučene su tangente na parabolu. Izračunati površinu figure ograničene parabolom i tangentama.

$y = x^2 - 4x + 5$ → $a = 1$, $b = -4$, $c = 5$ $\quad D = b^2 - 4ac = -4 < 0$

$\Rightarrow$ nema presjeka sa Oy osom

→ Tjeme $\left(-\frac{b}{2a}, -\frac{D}{4a}\right) \to \boxed{T(2, 1)}$

(*) Presjek sa Oy osom $\Rightarrow x = 0$

$$\begin{cases} y = x^2 - 4x + 5 \\ x = 0 \end{cases} \Rightarrow y = 5 \to \boxed{A(0,5)}$$

(*) Presjek parabole i prave

$$\begin{cases} y = x^2 - 4x + 5 \\ y = x + 1 \end{cases} \to \begin{array}{l} x + 1 = x^2 - 4x + 5 \\ x^2 - 5x + 4 = 0 \end{array}$$

$x_{1,2} = \frac{5 \pm \sqrt{25-16}}{2}$

$x_1 = 4$; $x_2 = 1$

$y_1 = 5$ $\quad y_2 = 2$

$\boxed{B_1(4,5)}$ $\quad \boxed{B_2(1,2)}$

(*) J-na tangente u tački $B_1(4,5)$

$t_1: y - 5 = y'(4)(x-4)$ → $t_1: y = 4x - 11$

$y' = 2x - 4$; $y'(4) = 4$

(*) J-na tangente u $B_2(1,2)$

$t_2: y - 2 = y'(1)(x-1)$

$t_2: y = -2x + 4$

(*) Presjek tangenti

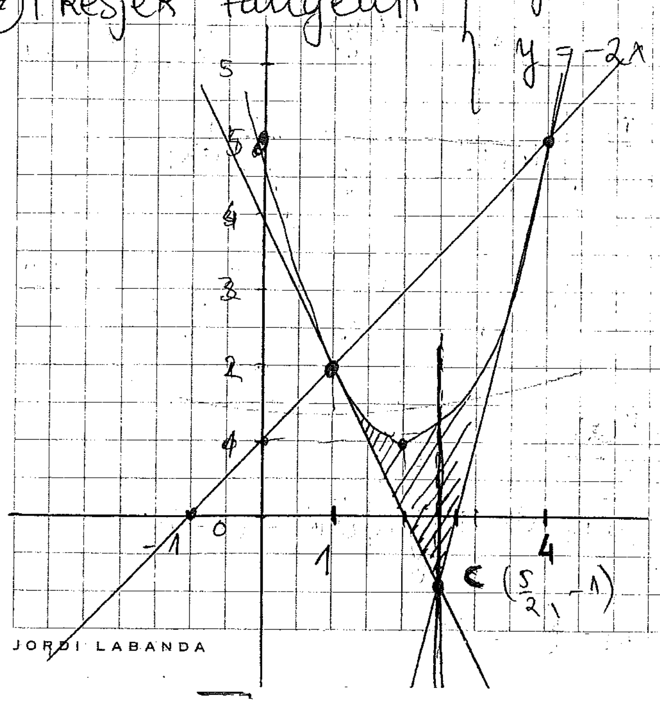
$$\begin{cases} y = 4x - 11 \\ y = -2x + 4 \end{cases} \to \boxed{C\left(\frac{5}{2}, -1\right)}$$

$y = 0 \Rightarrow -2x = -4$, $x = 2$

$y = 4x - 11$

$x = 0 \Rightarrow y = -11$

$y = 0 \Rightarrow 4x = 11 \to x =$



$$P = \int_1^{\frac{5}{2}} \text{parabola} - t_2 + \int_{5/2}^{4} \text{parab} - t_1$$

(85) Izračunati površinu ograničenu sa $y=2-|x|$ i $y=x^2$.

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

$$2-|x| = \begin{cases} 2-x, & x \geq 0 \\ 2+x, & x < 0 \end{cases}$$

Presjek parabole $y=x^2$ i prave $y=2-x$, za $x \geq 0$

$$\begin{cases} y = x^2 \\ y = 2-x \\ x \geq 0 \end{cases} \qquad x^2 = 2-x \qquad x^2+x-2=0 \qquad x_{1,2} = \frac{-1 \pm \sqrt{1+8}}{2} = \frac{-1 \pm 3}{2}$$

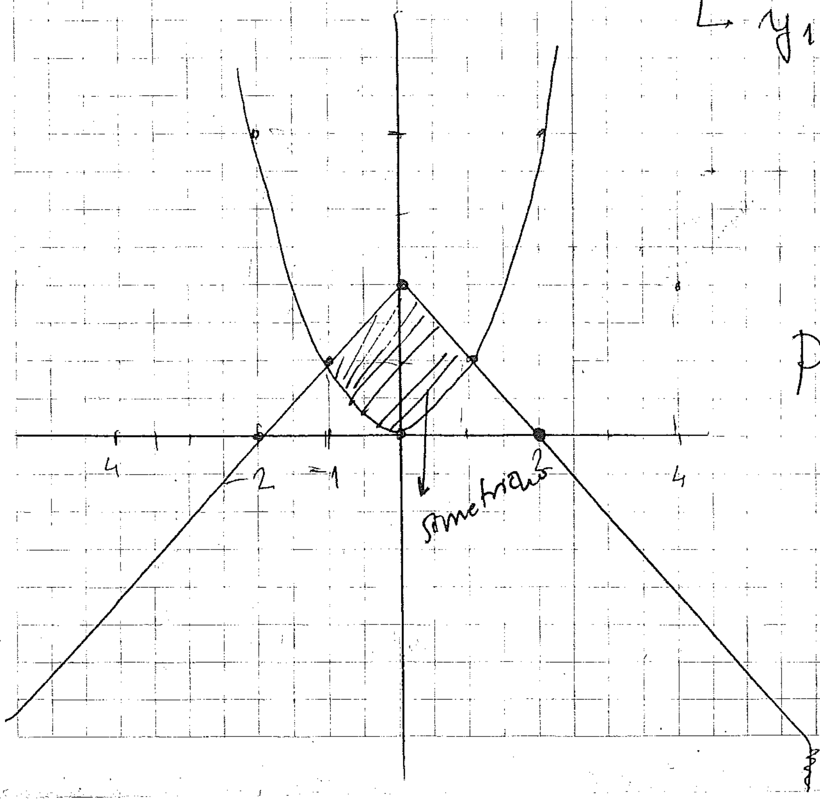
$\cancel{x_1 = -2}$ nije ≥ 0 ; $x_2 = 1 \Rightarrow y_2 = 1 \rightarrow \boxed{A(1,1)}$

Presjek parabole $y=x^2$ i prave $y=2+x$, za $x<0$

$$\begin{cases} y = x^2 \\ y = 2+x \\ x < 0 \end{cases} \qquad x^2 = 2+x \qquad x^2-x-2=0 \qquad x_{1,2} = \frac{+1 \pm \sqrt{1+8}}{2} = \frac{1 \pm 3}{2}$$

$x_1 = -1$ $\quad \cancel{x_2 = 2}$ nije <0

$\rightarrow y_1 = 1 \rightarrow \boxed{B(-1,1)}$



$y=0 \Rightarrow x=2$

$x=0 \quad y=2$

$y=0 \Rightarrow x=-2$

$$P = 2\int_{-1}^{0} (2+x-x^2)\,dx =$$

$$= 2\int_{0}^{1} (2-x-x^2)\,dx =$$

$= \ldots$

(86.) Izračunati površinu figure ograničene krivom $y=\frac{1}{x}$, njenom tangentom u tački $M(1,1)$ i pravom $x=3$.

Presjek krive $y=\frac{1}{x}$ i prave $x=3$

$$\begin{cases} y=\frac{1}{x} \\ x=3 \end{cases} \longrightarrow \boxed{A\left(3,\ \frac{1}{3}\right)}$$

J-na tangente u tački $M(1,1)$ je:

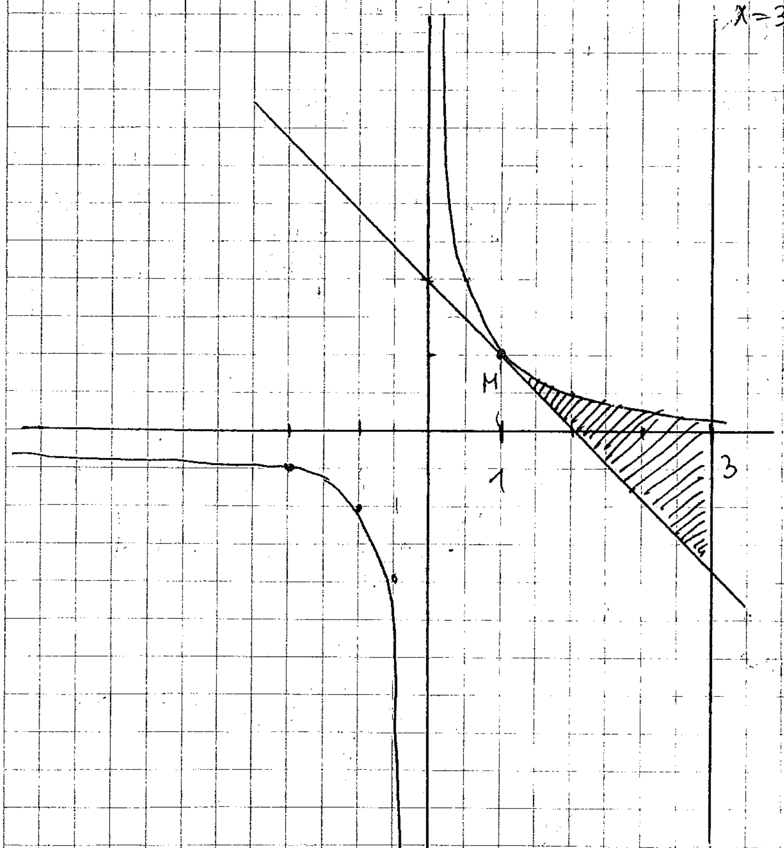
$$t:\ y-1=y'(1)(x-1) \longrightarrow y-1=-1(x-1)$$

$$y'=-\frac{1}{x^2}$$

$$y'(1)=-1$$

$$t:\ y=-x+2$$

$y=0 \Rightarrow x=2$
$x=0 \Rightarrow 2$



kriva-tangenta

$$P=\int_1^3\left(\frac{1}{x}+x-2\right)dx=$$

$$=\int_1^3\left(\frac{1}{x}+x-2\right)dx=$$

$$=\int_1^3\frac{1}{x}dx+\int_1^3 x\,dx-2\int_1^3 dx=$$

$$=\ln x\Big|_1^3+\frac{x^2}{2}\Big|_1^3-2x\Big|_1^3=$$

$$=\ln 3-\ln 1+\frac{3^2}{2}-\frac{1}{2}-6+2=$$

$l = \int_a \sqrt{1+(f'(x))^2}\,dx$

Dužina luka

(87) Naći dužinu luka koji na paraboli $y^2 = 2x+1$ odsijeca prava $x - y = 1 \to y = x-1$

Jednostavnije $x(y)$

$\to$ presjek sa ~~Oy osom~~ Ox osom

$y = 0 \to 2x+1=0 \to 2x = -1 \Rightarrow x = -\frac{1}{2}$ $\quad A(-\frac{1}{2}, 0)$

$2x = -y^2+1$

Tjeme $\to \left(-\frac{D}{4a}, -\frac{b}{2a}\right)$ $\quad x = -\frac{1}{2}y^2 + \frac{1}{2}$

$b=0 \Rightarrow$ presjek sa Ox osom tjeme $\to$ $\boxed{T(-\frac{1}{2}, 0)}$

$\to x=0 \Leftrightarrow y^2 = 1 \to y = \pm 1$ $\boxed{A_1(0,1)}$ $\boxed{A_2(0,-1)}$

presjek parabole i prave

$y^2 = 2x+1$

$x = +\frac{1}{2}y^2 + \frac{1}{2}$

$x = y+1$

$+\frac{1}{2}y^2 \cancel{+} \frac{1}{2} = y+1$

$\frac{1}{2}y^2 \cancel{+} y + \frac{1}{2} = 0 \,/\cdot 2$

$y_{1,2} =$

lakše:

$y^2 = 2x+1$

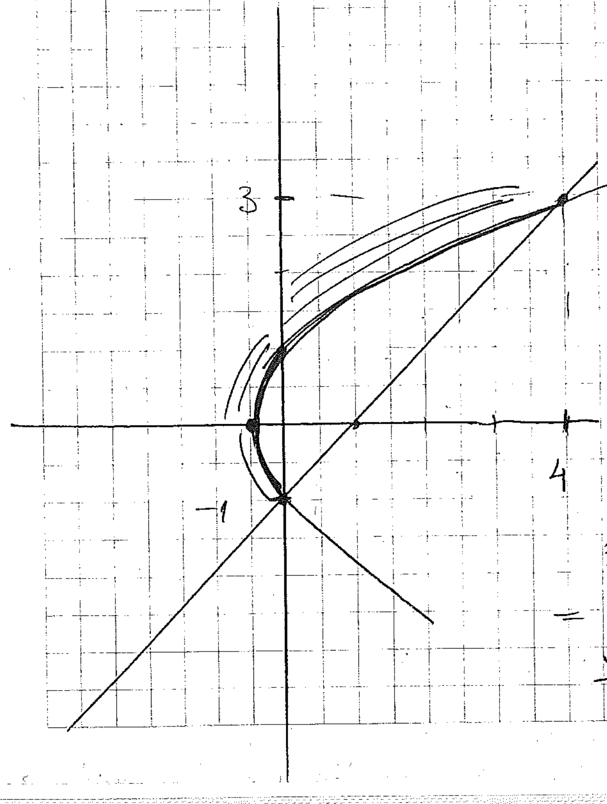
$y = x-1$

$(x-1)^2 = 2x+1$

$\hookrightarrow x_{1,2} = 0, 4$

$x = 0 \Rightarrow y = -1$ $\boxed{B_1(0,-1)}$

$x = 4 \Rightarrow y = 3$ $\boxed{B_2(4,3)}$



Jednostavnije

~~$y=f$~~ $x = f(y)$

~~$l = \int_{-\frac{1}{2}}$~~

$2x = y^2 - 1$

$x = \frac{1}{2}y^2 - \frac{1}{2}$ $\quad f'(y) = y$

$$l = \int_{-1}^{3} \sqrt{1+(f'(y))^2}\,dy =$$

$$= \int_{-1}^{3} \sqrt{1+y^2}\,dy$$

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$$l=\int_{-1}^{3}\sqrt{1+y^2}\,dy=\left[\begin{array}{l} U=\sqrt{1+y^2} \Rightarrow dU=\frac{1}{\cancel{2}}\frac{\cancel{2}y}{\sqrt{1+y^2}}dy \\ dV=dy \Rightarrow V=\int dy=y \end{array}\right]=$$

$$=\cancel{\int_{-1}^{3}}=y\cdot\sqrt{1+y^2}\Big|_{-1}^{3}-\int_{-1}^{3}\frac{y^2+1-1}{\sqrt{1+y^2}}\,dy=$$

$$=y\cdot\sqrt{1+y^2}\Big|_{-1}^{3}-\underbrace{\int_{-1}^{3}\sqrt{1+y^2}\,dy}_{l}+\int_{-1}^{3}\frac{dy}{\sqrt{1+y^2}}$$

$$l=3\sqrt{10}+\sqrt{2}-l+\ln\left|y+\sqrt{y^2+1}\right|\Big|_{-1}^{3}$$

$$2l=\ldots\quad;\quad l=(\ldots)\cdot\frac{1}{2}$$

---

(88) Naći dužinu luka krive $\begin{cases} x=\frac{t^6}{6} \\ y=2-\frac{t^4}{4} \end{cases}$

između presječnih tačaka sa koordinatnim osama

dužina luka parametarski $l=\int_{\alpha}^{\beta}\sqrt{(x'(t))^2+(y'(t))^2}\,dt$

Presjek sa Oy osom $\to x=0$ Presjek sa Ox osom $\to y=0$

$\frac{t^6}{6}=0 \to t=0 \to A(0,2)$ $\quad 2-\frac{t^4}{4}=0$

$y=2-\frac{0}{4}=2$ $\quad t^4=8 \to t=\sqrt[4]{8}$

$x'(t)=t^5$

$y'(t)=-t^3$ $\quad l=\int_{t_1}^{t_2}\sqrt{x'^2(t)+y'^2(t)}\,dt=\int_{0}^{\sqrt[4]{8}}\sqrt{t^{10}+t^6}\,dt=$

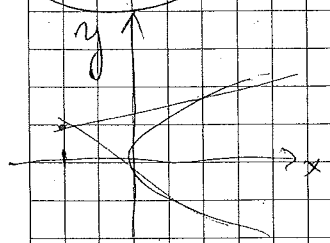
$$=\int_{0}^{\sqrt[4]{8}}t^3\sqrt{t^4+1}\,dt=\left[\begin{array}{l} t^4+1=z \\ 4t^3\,dt=dz \\ t^3dt=\frac{1}{4}dz \end{array}\quad \begin{array}{c|c|c} t & 0 & \sqrt[4]{8} \\ z & 1 & 9 \end{array}\right]=$$

$$=\frac{1}{4}\int_{1}^{9}\sqrt{z}\,dz=\frac{1}{4}\cdot\frac{2}{3}\sqrt{z^3}\Big|_{1}^{9}=\frac{26}{6}=\frac{13}{3}$$

89) Iz tačke $P(-2,1)$ van parabole $y^2=4x$ povučene su tangente na nju

a) Naći j-ne tih tangenti

b) Izračunati dužinu luka od tjemena parabole do one tačke dodira sa tangentom koja ima veću apscisu $\longrightarrow$ x koordinata



Neka je $y=kx+n$ j-na tangente iz tačke $P(-2,1)$ na parabolu $y^2=4x$,

Tačka P pripada tangenti pa je:

$$1 = k\cdot(-2)+n \rightarrow -2k+n=1$$

Nađimo presjek tangente i parabole

$$\begin{cases} y^2=4x \\ y=kx+n \end{cases} \quad (kx+n)^2=4x \quad k^2x^2+2kxn+n^2=4x \quad \rightarrow k^2x^2+2(kn-2)x+n^2=0$$

Kako tangenta dodiruje krivu u jednoj tački, to posledicja jna ima samo jedno rješ. tj $D=0$

$a=k^2$ $\quad D=b^2-4ac$

$b=2(kn-2)$ $\quad 4(kn-2)^2-4k^2n^2=0$

$c=n^2$ $\quad \cancel{k^2n^2}-4kn+4-\cancel{k^2n^2}=0$

$$kn=1$$

$$\begin{cases} -2k+n=1 \\ kn=1 \end{cases} (\cdot k) \rightarrow \begin{cases} -2kn+n=1 \\ 2k^2+k-1=0 \end{cases}$$

$$k_{1,2}=\frac{-1\pm\sqrt{1+8}}{4}; \quad k_{1,2}=\frac{-1\pm3}{4} \quad \begin{matrix} k_1=\frac{1}{2} & n_1=2 \\ k_2=-1 & n_2=-1 \end{matrix}$$

$$t_1: y=\frac{1}{2}x+2 \qquad t_2: y=-x-1$$

Tačka dodira parabole i t_1

$$\begin{cases} y^2 = 4x \\ y = \frac{1}{2}x + 2 \end{cases} \longrightarrow A(4,4)$$

prvi izvod f-je u tački

1. izvod -1

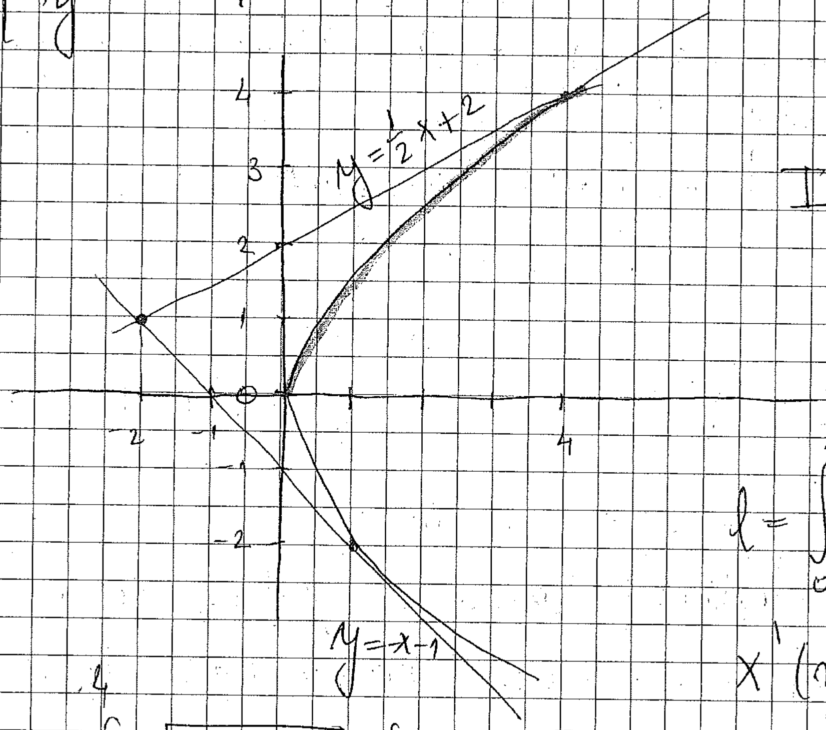
1. izvod $\frac{1}{2}$

Tačka dodira parabole i t_2

$$\begin{cases} y^2 = 4x \\ y = -x - 1 \end{cases} \longrightarrow B(1,-2)$$

$A(4,4) \rightarrow$ Apscisa 4

$B(1,-2) \rightarrow$ Apscisa 1



I x f-ja od y

$$x = \frac{y^2}{4}$$

$$l = \int_0^4 \sqrt{1 + x'^2(y)}\, dy$$

$$x'(y) = \frac{y}{2}$$

$$l = \int_0^4 \sqrt{1 + \frac{y^2}{4}}\, dy = \left[\begin{array}{l} \frac{y}{2} = t \\ dy = 2dt \end{array} \quad \begin{array}{c|c|c} y & 0 & 4 \\ \hline t & 0 & 2 \end{array} \right] =$$

$$= 2\int_0^2 \sqrt{1+t^2}\, dt$$ → već rađeno parcijalnom int

$$l = 2\sqrt{5} + \ln(2+\sqrt{5})$$

ZAPREMINA

90. Naći zapreminu tijela koje nastaje rotacijom figure ograničene krivom $y = 2x - x^2$ i pravom $y = 0$ (x osa) oko: a) Ox ose, b) Oy ose

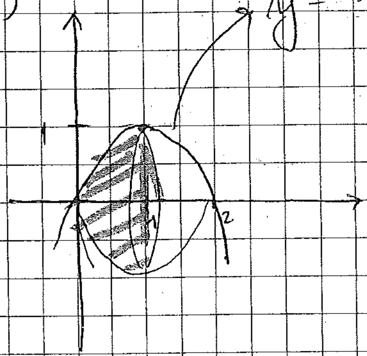
$y = -x^2 + 2x$

$y = 0 \Leftrightarrow x(2-x) = 0 \Leftrightarrow x = 0 \vee x = 2 \to O(0,0)\ A(2,0)$

$a = -1;\ b = 2;\ c = 0;\ D = 4;$ Tjeme $\left(-\frac{b}{2a}, \frac{-D}{4a}\right)$

$T\left(-\frac{2}{-2}, \frac{-4}{-4}\right) \longrightarrow T(1,1)$

a)



$y = -x^2 + 2x$

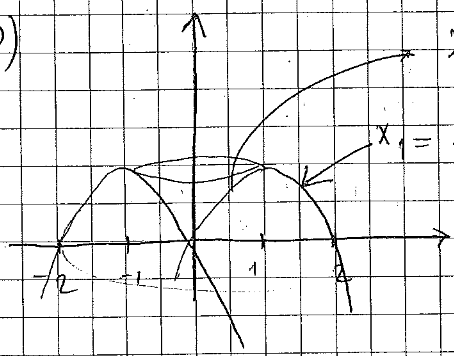
$V = 2\pi \int_0^1 y^2(x)\,dx$

$V = 2\pi \int_0^1 (2x - x^2)^2\,dx =$

$= 2\pi \int_0^1 (4x^2 - 4x^3 + x^4)\,dx =$

$= \frac{16}{15}\pi \longrightarrow$ a more i $V = \pi \int_0^2 y^2(x)\,dx$

b)



$x_2(y) = 1 - \sqrt{1-y}$

$x_1 = 1 + \sqrt{1-y}$

$y = 2x - x^2$

$x^2 - 2x + y = 0$

$x_{1,2} = \frac{2 \pm \sqrt{4-4y}}{2}$ $\qquad x_{1,2} = 1 \pm \sqrt{1}$

$x_{1,2} = \frac{2 \pm 2\sqrt{1-y}}{2}$

$x_{1,2} = 1 \pm \sqrt{1-y}$

$$V = \pi \int_0^1 \left(x_1^2(y) - x_2^2(y)\right) dy$$

$V = \pi \int_0^1 \left((1 + \sqrt{1-y})^2 - (1 - \sqrt{1-y})^2\right) dy$

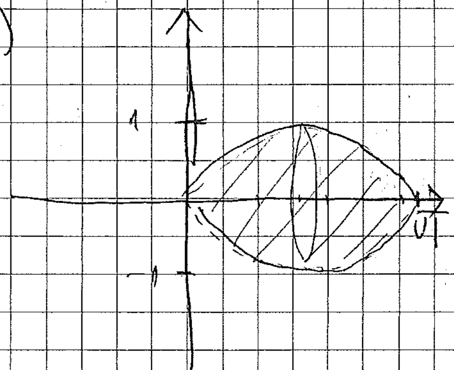
$V = \pi \int_0^1 \left(\cancel{1} + 2\sqrt{1-y} + \cancel{1} - \cancel{y} - \cancel{1} + 2\sqrt{1-y} \cancel{-1+y}\right) dy =$

$= \pi \cdot 4 \int_0^1 \sqrt{1-y}\,dy = \left| \begin{matrix} 1-y = t \\ dy = -dt \end{matrix} \quad \begin{matrix} y & 0 & 1 \\ t & 1 & 0 \end{matrix} \right| = \frac{8\pi}{3}$

$= -4\pi \int_1^0 \sqrt{t}\,dt = 4\pi \int_0^1 \sqrt{t}\,dt = 4\pi \cdot \frac{2}{3}\sqrt{t^3}\Big|_0^1$

(91) Naći zapreminu tijela koje nastaje rotacijom krive $y = \sin x$ i prave $y=0$ na segmentu $[0, \pi]$ oko a) Ox-ose; b) Oy ose

a)



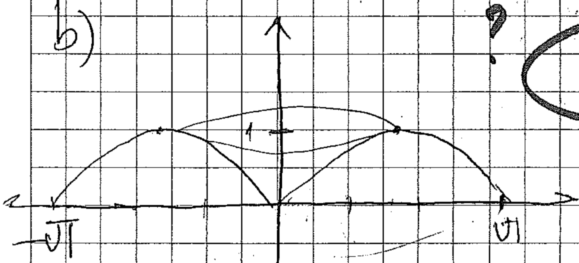
$$V = \pi \int_0^{\pi} \sin^2 x \, dx = \pi \int_0^{\pi} \frac{1-\cos 2x}{2} dx =$$

$$= \frac{\pi}{2} x \Big|_0^{\pi} - \frac{\pi}{2} \cdot \frac{1}{2} \sin 2x \Big|_0^{\pi} =$$

$$= \frac{\pi^2}{2}$$

$$V_y = 2\pi \int_a^b x f(x) \, dx$$

b)



?

$$\boxed{V_y = \pi \int_0^{\pi} x \sin x \, dx} = \text{parcijalno} \ldots =$$

$$= \pi \left(-x \cos x \Big|_0^{\pi} + \int_0^{\pi} \cos x \, dx \right)$$

$$= \pi \left(\pi + \sin x \Big|_0^{\pi} \right) = \pi^2$$

---

(92) U tački $P(3, y_0)$ parabole $y^2 = 2(x-1)$ povučena je tangenta. Izračunati zapreminu tijela koje nastaje rotacijom oko Ox ose, figure ograničene tangentom, parabolom i Ox osom.

$y^2 = 2x - 2$ → $a = \frac{1}{2}$; $b = 0$; $c = 1$; $D < 0$ → nema presjeka sa y osom

$2x = y^2 + 2$

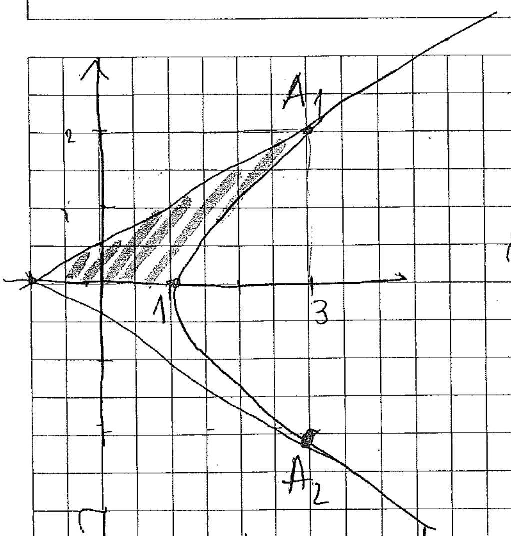
$x = \frac{1}{2} y^2 + 1$ $a > 0 \Rightarrow$ desno, ~~otvora~~

tjeme $\left(-\frac{D}{4a}, -\frac{b}{2a}\right) \to T\left(-\frac{-2}{2}, 0\right) \to T(1, 0)$ → presjek sa Ox osom $y=0$

P pripada paraboli

$y_0^2 = 2(3-1)$ $A_1(3, 2)$ → Zbog simetričnosti dovoljna tangenta u jednoj tački

$y_0^2 = 4 \to y_0 = \pm 2$ $A_2(3, -2)$



Tangenta u tački $A_1(3,2)$

$y-2 = y'(3)(x-3)$

$y^2 = 2x-2$

$2yy' = 2 \rightarrow y' = \frac{1}{y}$

$y'(A_1) = \frac{1}{2}$

J-na tangente $\rightarrow y-2 = \frac{1}{2}(x-3) : t_1$

$t_1 : y = \frac{1}{2}x + \frac{1}{2}$ $\qquad t_2 \rightarrow$ simetrično

$\rightarrow$ Zapremina ista kao kad rotira pola il cijela

$\rightarrow$ zapremina je razlika 2 zapremine:

$V = V_1 - V_2$

$$V = \pi \int_1^3 \left(\frac{1}{2}x + \frac{1}{2}\right)^2 dx - \pi \int_1^3 (2x-2)\,dx = \frac{4}{3}\pi$$

---

(93) Izračunati zapreminu nastalu rotacijom figure ograničene krivom $x^2+y^2=16$ i krivom $y^2=6x$:

a) oko Ox-ose

b) oko Oy-ose

J-na kruga sa centrom u $O(0,0)$ i poluprečnikom 4

* Presjek kruga i parabole

$$\begin{cases} x^2+y^2=16 \\ y^2 = 6x \end{cases} \rightarrow x^2+6x-16=0$$

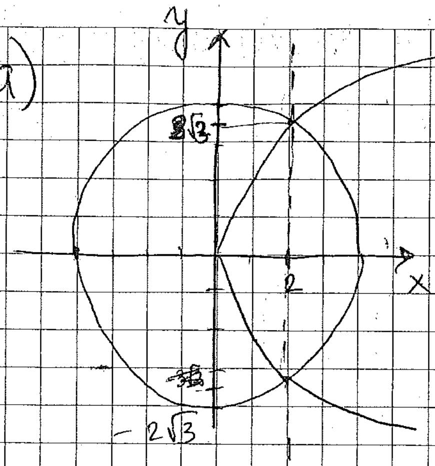
$$x_{1,2} = \frac{-6 \pm \sqrt{36+64}}{2} = \quad x_1 = \frac{-6+10}{2} = \boxed{2} \quad \cancel{x_2 = -8}$$

$\boxed{y^2 \geq 0}$ $\qquad y^2 = 6 \cdot x$, $x \geq 0$

$x_1 = 2 \rightarrow y = \pm\sqrt{12} = \pm 2\sqrt{3}$

$A_1(2, 2\sqrt{3}) \qquad A_2(2, -2\sqrt{3})$

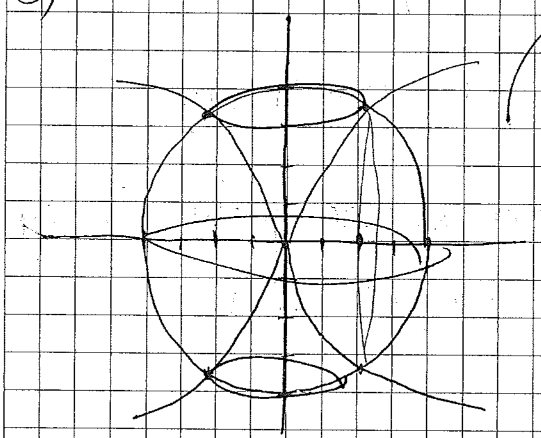
a)



$$V=\pi\int_0^2 6x\,dx+\pi\int_2^4(16-x^2)\,dx=$$

$$=\frac{76}{3}\pi$$

b)



$x^2+y^2=16 \rightarrow x^2=16-y^2$

$y^2=6x \rightarrow x=\frac{1}{6}y^2 \rightarrow x=\frac{1}{36}y^4$

$$V=\pi\int_0^{2\sqrt{3}}\Big(\underbrace{16-y^2}_{\text{dio kruga}}-\underbrace{\frac{1}{36}y^4}_{\text{parabola}}\Big)\,dy\cdot 2$$

$\cancel{V=}$ $V=\frac{224}{5}\sqrt{3}\pi$

$$V=\pi\left(16y-\frac{y^3}{3}-\frac{1}{36}\cdot\frac{1}{5}y^5\right)\Big|_0^{2\sqrt{3}}=$$

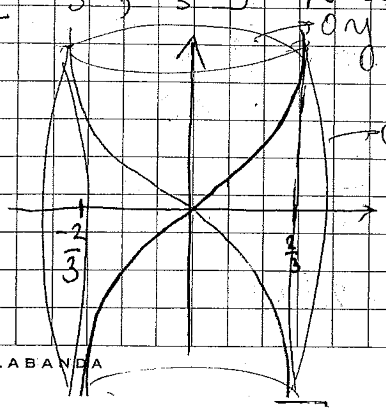
$$=\pi\left(16\cdot 2\sqrt{3}-\frac{2^3\cdot 3\cdot\sqrt{3}}{3}-\frac{1}{36}\cdot\frac{1}{5}\,2^5\cdot 3^2\sqrt{3}\right)=$$

$32-2^3-\frac{32\cdot 9}{36\cdot 5}=\frac{8}{5}$

$\frac{112}{5}\cdot 2\rightarrow \frac{224}{5}\sqrt{3}\pi$

9\. POVRŠINA ROTACIONOG TIJELA

94\. Luk krive $y=x^3$ rotira oko Ox ose na segmentu $\left[-\frac{2}{3},\frac{2}{3}\right]$. Naći površinu rotacionog tijela



$$P=2\cdot 2\pi\int_0^{2/3}y(x)\sqrt{1+y'^2(x)}\,dx,$$

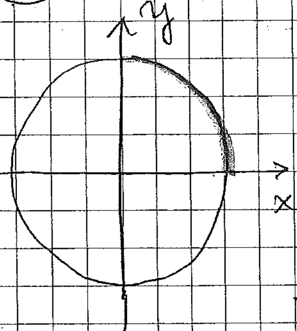
$y(x)=x^3;\quad y'(x)=3x^2$

$$P = 4\pi \int_0^{\frac{2}{3}} x^3 \cdot \sqrt{1+9x^4}\, dx = \left[\begin{array}{l} 1+9x^4 = t \\ x^3 dx = \frac{1}{36} dt \end{array} \quad \begin{array}{c|c|c} x & 0 & \frac{2}{3} \\ t & 1 & \frac{25}{9} \end{array} \right] =$$

$$= 4\pi \cdot \frac{1}{36} \cdot \int_1^{\frac{25}{9}} \sqrt{t}\, dt = \frac{2\pi}{27} \sqrt{t^3} \Big|_1^{\frac{25}{9}} = \frac{196}{729} \pi$$

95) Površina lopte poluprečnika r

$$x^2+y^2=r^2 \rightarrow y^2 = r^2-x^2 \rightarrow y = \pm\sqrt{R^2-x^2}$$



lopta nastaje rotacijom ovog kruga oko Ox ose

$\rightarrow$ svejedno da li rotira cio ili pola kruga $\rightarrow$ neka rotira $\frac{1}{4}$ pa množimo sa 2

$$P = 2 \cdot 2\pi \int_0^r y(x) \cdot \sqrt{1+y'^2(x)}\, dx$$

$$y(x) = \sqrt{r^2-x^2} \qquad y = \sqrt{R^2-x^2}$$

$$y'(x) = -\frac{x}{\sqrt{r^2-x^2}}$$

jer uzimamo + dio

$$P = 4\pi \int_0^r \sqrt{r^2-x^2} \cdot \sqrt{1+\frac{x^2}{r^2-x^2}}\, dx =$$

$$= 4\pi \int_0^r \cancel{\sqrt{r^2-x^2}} \cdot \frac{r}{\cancel{\sqrt{r^2-x^2}}}\, dx$$

$$P = 4\pi r \int_0^r dx = 4\pi r \cdot x \Big|_0^r = 4\pi r^2 = P$$

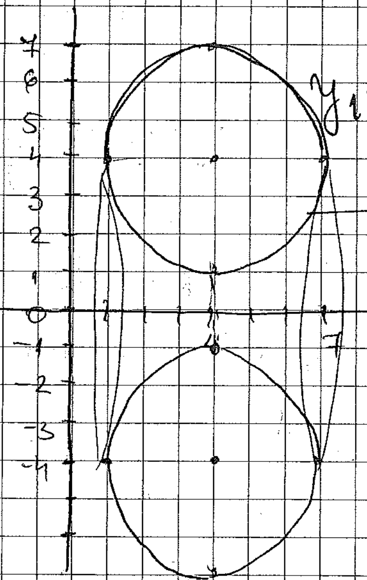
Kombinovano → kolokvijumi

(96) Izračunati površinu i zapreminu tijela koje se dobija rotacijom krive $x^2+y^2-8x-8y+23=0$ oko Ox ose.

$$x^2+y^2-8x-8y+23=0 \quad \Rightarrow (x-4)^2+(y-4)^2=9$$

$$(x-4)^2+(y-4)^2-9=0$$

Jednačina kruga sa centrom u tački $A(4,4)$ i poluprečnikom 3



$$y_1=4+\sqrt{9-(x-4)^2}$$

$$y_2=4-\sqrt{9-(x-4)^2}$$

$$V=\pi\int_1^7 \left(y_1^2(x)-y_2^2(x)\right)dx=$$

$$=\pi\int_1^7\left(\left(4+\sqrt{9-(x-4)^2}\right)^2-\left(4-\sqrt{9-(x-4)^2}\right)^2\right)dx=$$

$$=\pi\int_1^7\left(\cancel{16}+8\sqrt{9-(x-4)^2}+\cancel{9-(x-4)^2}-\cancel{16}+8\sqrt{9-(x-4)^2}-\cancel{\left(9-(x-4)^2\right)}\right)dx$$

$$=\pi\int_1^7 16\sqrt{9-(x-4)^2}\,dx=\left[\begin{matrix}x-4=t\\ dx=dt\end{matrix}\quad \begin{array}{c|c|c} x & 1 & 7\\ \hline t & -3 & 3\end{array}\right]=$$

$$=16\pi\int_{-3}^{3}\sqrt{9-t^2}\,dt=\left[\begin{matrix}t=3\sin z\\ dt=3\cos z\,dz\\ z=\arcsin\frac{t}{3}\end{matrix}\quad \begin{array}{c|c|c} t & -3 & 3\\ \hline z & -\frac{\pi}{2} & \frac{\pi}{2}\end{array}\right]=$$

$$=16\pi\cdot\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}3\cdot3\sqrt{1-\sin^2 z}\cos z\,dz=144\pi\cdot\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^2 z\,dz=$$

$$=144\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1+\cos 2z}{2}\,dz=144\pi\left(\frac{\pi}{2}\ldots\right)\quad \frac{1}{2}\left(\cos\frac{\pi}{2}-\cos\ldots\right)\quad \frac{72}{2}\quad 72\pi^2$$

Povrsina → gornji dio rotira + donji dio rotira

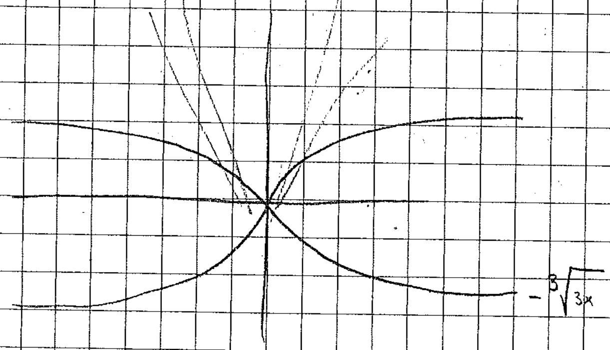
$$P = 2\pi \int_1^7 y_1(x) \cdot \sqrt{1+y_1'^2(x)}\, dx + 2\pi \int_1^7 y_2(x) \sqrt{1+y_2'^2(x)}\, dx$$

$$y_1'(x) = \frac{-(x-4)}{\sqrt{9-(x-4)^2}} \; ; \; y_2'(x) = \frac{x-4}{\sqrt{9-(x-4)^2}}$$

$P_1 =$

97. Figura F ograničena krivama $y=-\sqrt[3]{3x}$, $y=3|x|$ rotira oko Oy ose. Skicirati figuru F i izračunati površinu tijela koje nastaje rotacijom.

$$y=3|x| = \begin{cases} -3x, & x<0 \\ 3x, & x\geq 0 \end{cases}$$



→ presjek krive i prave

→ $y=-\sqrt[3]{3x}$, $y=3|x|$

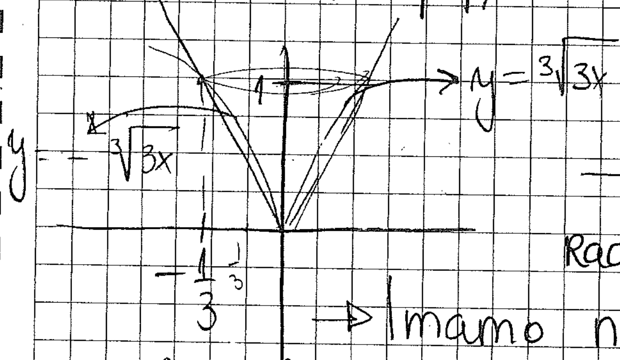
$$-\sqrt[3]{3x}=3|x| \;/^3 \to -3x = 3^3\cdot|x^3|$$

$$-x = 9|x^3| \;/:x$$

$$+1 = 9|x^2|$$

$$x^2=\frac{1}{9} \to x = \pm\frac{1}{3}$$

ali



→ Ako je fja sa kojom radimo negativna → !! apsolutna vr.

→ Imamo nešto što opisuje kriva i nešto što opisuje prava → $P=P_1+P_2$

$$y=\sqrt[3]{3x}$$

$$y^3=3x$$

$$x=\frac{1}{3}y^3$$

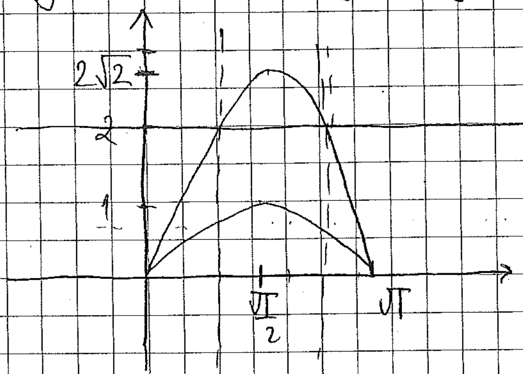
$$x'=\frac{1}{\cancel{3}}\cancel{3}y^2$$

$$P_1=\int_0^1 x(y)\cdot\sqrt{1+x'^2(y)}\,dy$$

$$P_1=\int_0^1 \frac{1}{3}y^3\cdot\sqrt{1+y^4}\,dy = \frac{1}{3}\int_0^1 y^3\sqrt{1+y^4}\,dy = \left| \begin{array}{l} 1+y^4=t \\ 3y^3\,dy=dt \\ y^3\,dy=\frac{1}{3}dt \end{array} \right|$$

(98) Figura ograničena lukovima krivih $y = \sin x$ i $y = 2\sqrt{2}\sin x$ na intervalu $[0, \pi]$ i pravom $y = 2$ rotira oko Ox ose. Izračunati zapreminu tako dobijenog rotacionog tijela.

→ presjek



dovršiti kući :D

NESVOJSTVENI INTEGRALI

99. a) $\int_0^{+\infty} x \cdot e^{-2x} dx = \lim_{B \to +\infty} \int_0^B x e^{-2x} dx = \left[\begin{array}{l} x = U \to dU = dx \\ V = \int e^{-2x} dx = -\frac{1}{2} e^{-2x} \end{array} \right]$

$= -\frac{1}{2} \lim_{B \to +\infty} \left(x e^{-2x} \Big|_0^B - \int_0^b e^{-2x} dx \right) =$

$= -\frac{1}{2} \lim_{B \to +\infty} \left(\frac{x}{e^{2x}} \Big|_0^B + \frac{1}{2} \left(\frac{1}{e^{2B}} - 1 \right) \right) = \frac{1}{4}$

$0 \to e^{2x} \to$ brže raste od $y = x$

b) $\int_1^{+\infty} \frac{dx}{x\sqrt{1+x^2}} = \lim_{B \to +\infty} \int_1^B \frac{dx}{x\sqrt{1+x^2}} = \left[\begin{array}{l} \frac{1}{x} = t \to x = \frac{1}{t} \\ \to dx = -\frac{1}{t^2} \\ x^2 = \frac{1}{t^2} \end{array} \right]$

$\to \begin{array}{c|c|c} x & 1 & B \\ \hline t & 1 & \frac{1}{B} \end{array}$

$= -\lim_{B \to +\infty} \int_1^{\frac{1}{B}} \frac{\frac{1}{t^2}}{\frac{1}{t} \sqrt{1 + \frac{1}{t^2}}} dt =$

$= -\lim_{B \to +\infty} \int_1^{\frac{1}{B}} \frac{dt}{\sqrt{t^2+1}} = -\lim_{B \to +\infty} \left(\ln \left| t + \sqrt{t^2+1} \right| \right) \Big|_1^{\frac{1}{B}} =$

$= -\lim_{B \to +\infty} \left(\underbrace{\ln \left| \frac{1}{B} + \sqrt{\frac{1}{B^2} + 1} \right|}_{\ln 1 = 0} - \ln \left| 1 + \sqrt{1+1} \right| \right) =$

$= \cancel{-\lim_{B \to +\infty}} = \ln(1 + \sqrt{2})$

100. $\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \lim_{B \to 1^-} \int_0^B \frac{dx}{\sqrt{1-x^2}} = \lim_{B \to 1^-} \arcsin x \Big|_0^B =$

$= \lim_{B \to 1^-} (\arcsin B - \underbrace{\arcsin 0}_{0}) = \frac{\pi}{2}$

101.

$$\int_2^3 \frac{dx}{\sqrt{6x-x^2-8}} = \left[\; D = 36-32 = 4 \to \quad x_{1,2} = \frac{-6\pm 2}{-2} \begin{matrix} \nearrow 4 \\ \searrow 2 \end{matrix} \quad -(x-2)(x-4) \;\right] =$$

$$= \int_2^3 \frac{dx}{\sqrt{-(x-2)(x-4)}} = \lim_{A\to 2^+} \int_a^3 \frac{dx}{\sqrt{-(x-2)(x-4)}} = \left[\; 1-(x-3)^2 \;\right]$$

$$= \lim_{a\to 2^+} \int_a^3 \frac{dx}{\sqrt{1-(x-3)^2}} = \lim_{a\to 2^+} \left(\arcsin(x-3)\Big|_a^3 \right) =$$

$$= \lim_{a\to 2^+} \left(\underbrace{\arcsin 0}_{0} - \arcsin(a-3) \right) = -\left(-\frac{\pi}{2}\right) = \frac{\pi}{2}$$

102.

problem u tački 0 → zato nesvojstveni

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos x}{\sqrt[3]{\sin^2 x}}\, dx = \int_{-\frac{\pi}{2}}^{0} \frac{\cos x}{\sqrt[3]{\sin^2 x}}\, dx + \int_{0}^{\frac{\pi}{2}} \frac{\cos x}{\sqrt[3]{\sin^2 x}}\, dx =$$

$$= \lim_{B\to 0^-} \int_{-\frac{\pi}{2}}^{B} \frac{\cos x}{\sqrt[3]{\sin^2 x}}\, dx + \lim_{A\to 0^+} \int_{A}^{\frac{\pi}{2}} \frac{\cos x}{\sqrt[3]{\sin^2 x}}\, dx = \left[\; \begin{matrix} \sin x = t \\ \cos x\, dx = dt \end{matrix} \quad \sin^{-\frac{2}{3}} x \;\right] =$$

$$= \lim_{B\to 0^-} 3\sqrt[3]{\sin x}\,\Big|_{-\frac{\pi}{2}}^{B} + \lim_{A\to 0^+} 3\sqrt[3]{\sin x}\,\Big|_{A}^{\frac{\pi}{2}} =$$

$$= -(-3) + 3 = 6$$

103. Ispitati konvergenciju integrala

$$\int_0^1 \frac{dx}{x^{\alpha}}$$

$\alpha \neq 1$: $\int_0^1 \frac{dx}{x^\alpha} = \lim_{a \to 0^+} \int_a^1 x^{-\alpha} dx = \lim_{a \to 0^+} \frac{x^{-\alpha+1}}{1-\alpha} \Big|_a^1 =$

$= \lim_{a \to 0^+} \left(\frac{1}{1-\alpha} - \frac{a^{1-\alpha}}{1-\alpha} \right) = \begin{cases} +\infty, & \alpha > 1 \\ \frac{1}{1-\alpha}, & \alpha < 1 \end{cases}$

$\alpha > 1 \to 1-\alpha < 0 \to$ ~~a~~ $\frac{1}{a^{\alpha-1}}$

~~a~~ $\frac{1}{0} \to \infty$

$\alpha < 1 \to 1-\alpha > 0 \to \frac{\overset{0}{a^{1-\alpha}}}{1-\alpha} \to 0 \to \frac{1}{1-\alpha}$

$\alpha = 1$: $\int_0^1 \frac{dx}{x} = \lim_{a \to 0^+} \int_a^1 \frac{dx}{x} = \lim_{a \to 0^+} \ln|x| \Big|_a^1 =$

$= \lim_{a \to 0^+} (\underset{0}{\ln 1} - \underset{-\infty}{\ln a}) = +\infty$

→ Za $\alpha < 1$ → Integral konvergira i njegova vrijednost je $\frac{1}{1-\alpha}$, a za $\alpha \geq 1$ integral diverg.

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(104) Ispitati konvergenciju

$\int_1^{+\infty} \frac{dx}{x^\alpha}$

$\alpha \neq 1$: $\int_1^{+\infty} \frac{dx}{x^\alpha} = \lim_{B \to +\infty} \frac{x^{1-\alpha}}{1-\alpha} \Big|_1^B = \lim_{B \to +\infty} \left(\frac{B^{1-\alpha}}{1-\alpha} - \frac{1^{1-\alpha}}{1-\alpha} \right)$

$= \begin{cases} -\frac{1}{1-\alpha}, & \alpha > 1 \\ +\infty, & \alpha < 1 \end{cases}$

$\alpha > 1 \Rightarrow 1-\alpha < 1 \to \frac{1}{B^{\alpha-1}} \xrightarrow{B \to \infty} 0$

$\alpha < 1 \Rightarrow 1-\alpha > 1 \xrightarrow{B \to \infty} B^{1-\alpha} \xrightarrow{B \to \infty} \infty$

$\alpha = 1$ $\int_1^{+\infty} \frac{dx}{x^\alpha} = \lim_{B \to \infty} \int_1^B \frac{dx}{\cancel{x}} = \lim_{B \to \infty} \ln x \Big|_1^B =$

$= \lim_{B \to +\infty} (\underset{\infty}{\ln B} - \underset{0}{\ln 1}) = \infty$

→ Za $\alpha > 1$ Integral konverg, a za $\alpha \leq 1$ diverg.

(105.) Ispitati konvergenciju integrala

$$\int_0^{+\infty} \frac{\sin x \cdot \operatorname{arctg} x}{1+x^2}\,dx$$

$$\int_{+1}^{\infty} \frac{dx}{\sqrt{e^x x}}$$

Neka je $f(x) = \dfrac{\sin x \cdot \operatorname{arctg} x}{1+x^2}$

$$|f(x)| = \left| \frac{\sin x \cdot \operatorname{arctg} x}{1+x^2} \right| \leq \frac{1 \cdot \frac{\pi}{2}}{1+x^2}, \quad \forall x \geq 0$$

$$|f(x)| \leq g(x), \quad g(x) = \frac{\pi}{2} \cdot \frac{1}{1+x^2}$$

$$\int_0^{+\infty} g(x)\,dx = \int_0^{+\infty} \frac{\pi}{2} \cdot \frac{1}{1+x^2}\,dx = \frac{\pi}{2} \lim_{B \to +\infty} \int_0^B \frac{dx}{1+x^2} =$$

$$= \frac{\pi}{2} \cdot \lim_{B \to +\infty} (\operatorname{arctg} B - \operatorname{arctg} 0) = \frac{\pi^2}{4}$$

$\Rightarrow \displaystyle\int_0^{+\infty} g(x)\,dx$ konvergira

$$\left.\begin{array}{l} |f(x)| \leq g(x), \quad \forall x \geq 0 \\ \displaystyle\int_0^{+\infty} g(x)\,dx \text{ konvergira} \end{array}\right\} \Rightarrow \int_0^{+\infty} f(x)\,dx \text{ takođe konvergira}$$